Méthodes de traitement des couches limites pour la résolution d'équations de transport (avec une approche de Trefftz-DG)

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Our target : numerics for transport equations

- Transport equations

$$\partial_t I(t, \mathbf{x}, \mathbf{\Omega}) + \mathbf{\Omega} \cdot \nabla I(t, \mathbf{x}, \mathbf{\Omega}) = -\sigma_a(\mathbf{x}) I(t, \mathbf{x}, \mathbf{\Omega}) + \sigma_s(\mathbf{x}) \left(|I| - I(t, \mathbf{x}, \mathbf{\Omega}) \right),$$

Coefficients σ_a, σ_s a often **constant per sud-domains but discontinuous** (neutrons, radiation, ...)

- Need simplified reduced models.

- The S_n model is

$$\partial_t u_i + \mathbf{a}_i \cdot \nabla u_i = \sigma_s(\mathbf{x}) \left(\widehat{u} - u_i \right) - \sigma_a(\mathbf{x}) u_i, \quad 1 \leq i \leq n,$$

where $\widehat{u} = \frac{u_1 + \cdots + u_n}{n}$ is the mean value.

- The P_n model writes : let $I_p = \int I(t, \mathbf{x}, \mathbf{\Omega}) \mathbf{\Omega}^p d\Omega$

$$\begin{cases} \partial_t I_p + \nabla \cdot I_{p+1} = -(\sigma_a + \sigma_s)(\mathbf{x}) I_p + \delta_{p0} \sigma_s(\mathbf{x}) I_0, \quad 0 \leq p \leq n \\ I \in P_n(\mathbf{\Omega}). \end{cases}$$

References (1926 Trefftz seminal paper)

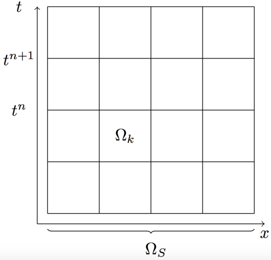

Trefftz methods for time harmonic wave problems :
- PUM (Partition of Unity Method) : (Melenk, Babuska, 96')
- UWVF (Ultra Weak Variational Formulation) : D. (94'), Cessenat-D. (98'), Monk-Huttunen and al (a series, last one 14'), Gittelson-Hiptmair-Perugia (09'), . . .
- DG (Discontinuous Galerkin), Weak DG : Monk-Buffa (08'), Hiptmair-Perugia (09'), Hiptmair-Moiola-Perugia(94), . . .
- GPW (Generalized Plane Waves) : Imbert-Gérard-D., Imbert-Gérard (15'), . . .
- Enrichement : Farhat and al (ex : The discontinuous enrichment method for multiscale analysis 03'), (14'),
- Trefftz : Hiptmair-Moiola-Perugia (15'),

Growing interest of Trefftz methods for time dependent problems :
- Macia-Sokala (11'), Trefftz on a single element
- Petersen-Farhat-Tezaur, Wang (14'), DG with Lagrange multipliers
- Egger-Kretzschmar-Schnepp-Tsukerman-Weiland, (15'), Maxwell equations (1D)
- Kretzschmar-Moiola-Perugia-Schnepp (15'), analysis
- Farhat (14'), space-time special DG

Friedrichs systems with relaxation

Unknown is $\mathbf{u} \in \mathbb{R}^n$: for time domain problems, $A_0 = I_\mathrm{n}$ and $x_0 = t$

$$\begin{cases} \sum_{i=0}^{d} A_i \partial_i \mathbf{u} = -R(\mathbf{x})\mathbf{u}, & \text{in } \Omega, \\ M^- \mathbf{u} = M^- \mathbf{g}, & \text{in } \partial\Omega, \end{cases}$$



Example $n = 2$ and $d = 1$

$$\begin{cases} \partial_t p + \frac{c}{\varepsilon} \partial_x v = -\sigma_a p, \\ \partial_t v + \frac{c}{\varepsilon} \partial_x p = -(\sigma_a + \frac{\sigma_s}{\varepsilon^2}) v, \end{cases}$$

Define the finite dimensional broken polynomial space of polynomials of d variables, of total degree at most q

$$P_h(\mathcal{T}_h) := \{\mathbf{v} \in L^2(\Omega), \mathbf{v}_{|\Omega_k} \in \mathbb{P}_q^d \ \forall \Omega_k \in \mathcal{T}_h\} \subset H^1(\mathcal{T}_h).$$

Definition

The standard DG method for Friedrichs systems is

$$\begin{cases} \textit{find } \mathbf{u}_h \in P_h(\mathcal{T}_h) \textit{ such that} \\ a_{DG}(\mathbf{u}_h, \mathbf{v}_h) = l(\mathbf{v}_h), \quad \forall \mathbf{v}_h \in P_h(\mathcal{T}_h). \end{cases} \tag{1}$$

- By construction, the scheme is implicit.
- It can be solved one time step after another.
- It can be run explicit.

- Ern, Guermond : Discontinuous Galerkin methods for Friedrichs' systems, 2006.
- Guermond, Kanschat, Ragusa : Discontinuous Galerkin for the radiative transport equation, 2014.
- Cheng, Shu : High order positivity-preserving DG methods for radiative transfer equations, 2016.

Motivations
Trefftz-DG
More numerical results

For $L = \sum_i A_i \partial_x + R(x)$, take a Trefftz space

$$V(\mathcal{T}_h) = \{\mathbf{v} \in H^1(\mathcal{T}_h), L\mathbf{v}_k = \mathbf{0} \quad \forall \Omega_k \in \mathcal{T}_h\} \subset H^1(\mathcal{T}_h).$$

For all $\mathbf{u}, \mathbf{v} \in V(\mathcal{T}_h)$: one has $\int_{\Omega_k} \left(L^* \mathbf{v}_k\right)^T \mathbf{u}_k = 2 \int_{\Omega_k} \mathbf{v}_k^T R \mathbf{u}_k$; we set

$$a_T(\mathbf{u}, \mathbf{v}) = -\sum_k \sum_{j<k} \int_{\Sigma_{kj}} (M_{kj}^- \mathbf{v}_k + M_{kj}^+ \mathbf{v}_j)^T (\mathbf{u}_k - \mathbf{u}_j) - \sum_k \int_{\Sigma_{kk}} \mathbf{v}_k^T M_k^- \mathbf{u}_k.$$

Definition

Assume $V_h(\mathcal{T}_h)$ is a finite subspace of $V(\mathcal{T}_h)$. The TDG method is

$$\begin{cases} \text{find } \mathbf{u}_h \in V_h(\mathcal{T}_h) \text{ such that} \\ a_T(\mathbf{u}_h, \mathbf{v}_h) = l(\mathbf{v}_h), \quad \forall \mathbf{v}_h \in V_h(\mathcal{T}_h). \end{cases} \tag{2}$$

TDG basis : a simple example

- Consider the stationary P_1 model in one dimension

$$\begin{cases} \frac{c}{\varepsilon}\partial_x v = -\sigma_a p, \\ \frac{c}{\varepsilon}\partial_x p = -\sigma_t v, \quad \sigma_t = \sigma_a + \frac{\sigma_s}{\varepsilon^2}. \end{cases}$$

The unknown is $\mathbf{u} = (p, v)^T$.

- Take solutions as $\mathbf{z}e^{\lambda x}$ one gets λ by solving $\det(A_1\lambda + R) = 0$:

$\mathbf{e}_1(x) = \begin{pmatrix} -\sqrt{\sigma_t} \\ \sqrt{\sigma_a} \end{pmatrix} e^{\frac{\varepsilon}{c}\sqrt{\sigma_a\sigma_t}x}$, $\mathbf{e}_2(x) = \begin{pmatrix} \sqrt{\sigma_t} \\ \sqrt{\sigma_a} \end{pmatrix} e^{-\frac{\varepsilon}{c}\sqrt{\sigma_a\sigma_t}x}$.

In this case, $\mathbf{e}_{1,2} \notin P_1(x)^2$ and $Span(V_h) = \{\mathbf{e}_1, \mathbf{e}_2\} \not\subset P_1(x)^2$.

- If $\sigma_a = 0$, there is a degeneracy.

Take $\mathbf{e}_1(x) = (1, 0)^T$ and $\mathbf{e}_2(x) = (-\varepsilon\sigma_t x, c)$.

In this case, $\mathbf{e}_{1,2} \in P_1(x)^2$ and $Span(V_h) = \{\mathbf{e}_1, \mathbf{e}_2\} \subset P_1(x)^2$.

$$\begin{cases} \frac{c}{\varepsilon}\nabla \cdot \mathbf{v}(t,\mathbf{x}) = -\sigma_a(\mathbf{x})p(t,\mathbf{x}), \\ \frac{c}{\varepsilon}\nabla \mathbf{p}(t,\mathbf{x}) = -\sigma_t(\mathbf{x})\mathbf{v}(t,\mathbf{x}), \end{cases} \tag{3}$$

with the unknown $\mathbf{u} = (p, \mathbf{v})^T \in \mathbb{R}^3$.

$$A_1 = \frac{c}{\varepsilon}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ A_2 = \frac{c}{\varepsilon}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ R(\mathbf{x}) = \begin{pmatrix} \sigma_a(\mathbf{x}) & 0 & 0 \\ 0 & \sigma_t(\mathbf{x}) & 0 \\ 0 & 0 & \sigma_t(\mathbf{x}) \end{pmatrix}$$

Proposition (A first family of basis functions)

Take $\mathbf{d}_k = (\cos(\phi_k), \sin(\phi_k))^T \in \mathbb{R}^2$, $c \neq 0$ *and assume constant coefficients* σ_a, σ_t. *Consider*

$$\mathbf{e}_k = \begin{pmatrix} \sqrt{\sigma_t} \\ -\sqrt{\sigma_a}\mathbf{d}_k \end{pmatrix} e^{\frac{\varepsilon}{c}\sqrt{\sigma_a\sigma_t}(\mathbf{d}_k,\mathbf{x})}. \tag{4}$$

TDG : h-convergence

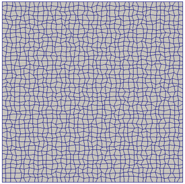
Proposition (Convergence in the dominant absorption regime : $\varepsilon = 1,\ \sigma_a > 0,\ \sigma_s \geq 0$)

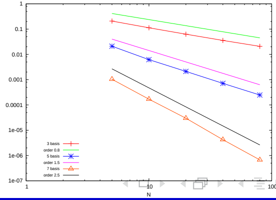
Consider $p_{\mathrm{TDG}} = 2n+1$ *basis functions*

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} \leq C h^{n-1/2} \|\mathbf{u}\|_{W^{n+1,\infty}(\Omega)},$$

with $h = \max_{\Omega_k \in \mathcal{T}_h} h_k,\ h_k = diam(\Omega_k)$.

order	1/2	3/2	5/2	7/2	9/2
p_{TDG}	3	5	7	9	11
p_{DG}	3	9	18	30	45





Diffusion asymptotic regime

For the P_1 model, the diffusion AP regime is

$$\partial_t p - \nabla \cdot \left(\frac{1}{\sigma_s(\mathbf{x})} \nabla p \right) = 0.$$

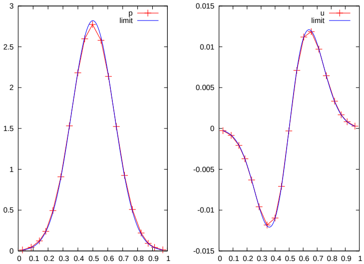
At the level of principles, TDG capture (in the cell) AP regimes by construction.

AP

In 1D, TDG with 2 basis functions reduces to

$$\frac{p_k^{n+1}-p_k^n}{\Delta t} + \frac{c}{2\varepsilon h}\Big[- p_{k+1} + 2p_k - p_{k-1} + (1-a)(v_{k+1} - v_{k-1})\Big]^{n+1} = 0,$$
$$(1+\frac{a^2}{3})\frac{v_k^{n+1}-v_k^n}{\Delta t} + \frac{c}{2\varepsilon h}\Big[a^2(v_{k+1}+2v_k+v_{k-1}) + (-v_{k+1}+2v_k-v_{k-1})$$
$$+(1+a)(p_{k+1}-p_{k-1})\Big]^{n+1} = -\frac{\sigma_s}{\varepsilon^2} v_k^{n+1},$$

It is an new scheme ($\neq$ Jin-Levermore, $\neq$ Gosse-Toscani). We proved it is AP using formal Hilbert expansion.

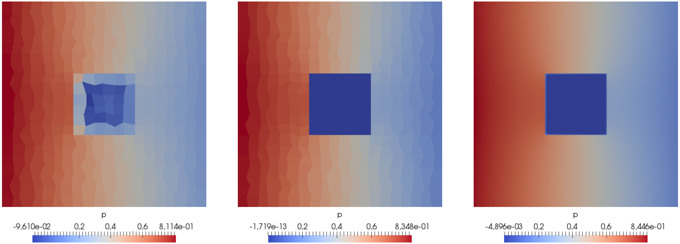


Random mesh with 20 nodes $dt = 0.01/20$.

- In 2D, no proof, but numerical tests $\Longrightarrow$ TDG yields diffusion-AP schemes.

P_1 : boundary layer

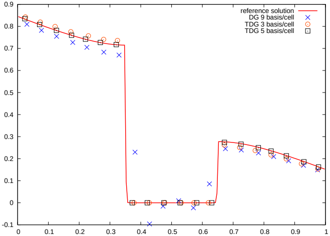
Exterior : $\sigma_s = 2$ and $\sigma_a = 0$. Interior : $\sigma_s = 10^5$ and $\sigma_a = 2$.

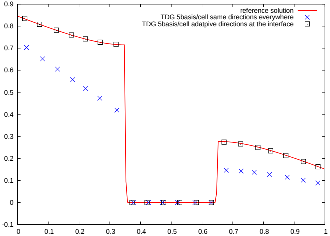


From left to right :
not shown : DG scheme with 3 basis functions per cell=FV,
DG scheme with 9 basis functions per cell,
TDG scheme with 3 basis functions per cell,
TDG scheme with 5 basis functions per cell
and reference solution.
For the TDG method the 4 directions at the interface are locally adapted.

Cut

One dimensional representation of the variable p at $y = 0.5$.





Left : comparison between the DG method with 9 basis/cell (cross), the TDG method with 3 basis/cell (circle) and the TDG method with 5 basis/cell (square). In both cases the directions at the interface in Ω_1 are locally adapted into the 4 directions.

Right : TDG method with 5 directions only (cross) versus TDG method where the 4 directions at the interface are locally adapted.

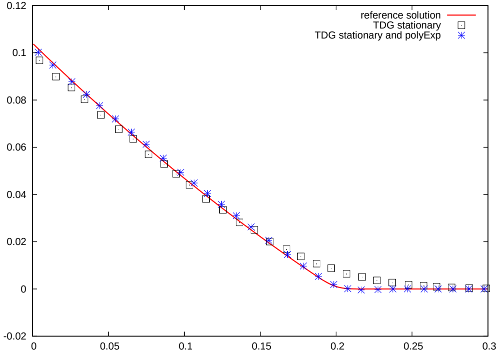
P_1/1D/non stationary

Motivations

Trefftz-DG

More numerical results

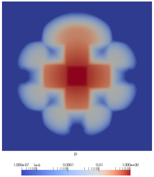
Idea : add basis functions with time dependance
Particles injected on the left



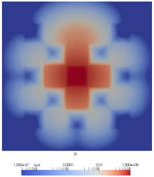
Accuracy at the foot of the wave much better with time dependent basis functions.

P_1/2D/non stationary

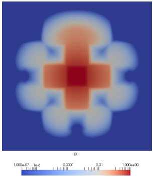
Time dependent test case from Brunner : source in the center.
Additional basis functions : 1 for the source + time dependent functions.



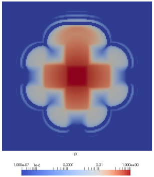
ref=DG



TDG stat. basis functions



TDG full set basis functions



TDG full modified set basis functions

Logscale, 140×140 cells, $t = 0.02$

P_n : general case

The stationary P_N (N odd) model writes

$$(15)\qquad \Big(A_1\partial_x + A_2\partial_y\Big)\mathbf{u}(t,\mathbf{x}) = -R\mathbf{u}(t,\mathbf{x}),$$

with $\mathbf{u} \in \mathbb{R}^m$, $A_1, A_2, R \in \mathbb{R}^m$, $\sigma_t = \varepsilon^2\sigma_a + \sigma_s$. The matrix $R := \frac{1}{\varepsilon^2}\sigma_t I_m - \frac{\sigma_s}{\varepsilon^2}\mathbf{e}_1\mathbf{e}_1^T$ is a diagonal matrix. Moreover the matrices A_1, A_2 and R have the following block structure [Hermeline, 2016]

$$(16)\qquad A_1 = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}, \quad R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix},$$

where A, $B \in \mathbb{R}^{m_1 \times m_2}$ are rectangular matrix and $R_1 \in \mathbb{R}^{m_1 \times m_1}$, $R_2 \in \mathbb{R}^{m_2 \times m_2}$ are diagonal matrix.

Proposition

Let $\sigma_t > 0$. Assume the matrix $(AA^T)^{-1}R_1$ admits $\mathbf{v}_1, ..., \mathbf{v}_{m_1} \in \mathbb{R}^{m_1}$ eigenvectors associated with the eigenvalues $\mu_1, ..., \mu_{m_1}$. Let $\mathbf{w}_i = -\varepsilon\sqrt{\frac{\mu_i}{\sigma_t}}A^T\mathbf{v}_i \in \mathbb{R}^{m_2}$, $\mathbf{z}_i = (\mathbf{v}_i^T, \mathbf{w}_i^T)^T \in \mathbb{R}^m$ and $\mathbf{d}_k = (\cos\theta_k, \sin\theta_k)^T \in \mathbb{R}^2$. Then the following functions are solution to the P_N model (15)

$$(\mathbf{e}_i)_k(\mathbf{x}) = U_{\theta_k}\mathbf{z}_i e^{\frac{1}{\varepsilon}\sqrt{\sigma_t\mu_i}(\mathbf{d}_k,\mathbf{x})}, \quad i = 1, ..., m_1,$$

where U_{θ_k} is an orthogonal matrix.

P_3 : basis functions

Example

For the P_3 model one has $m_1 = 4$ and denoting $\lambda_i = \frac{1}{\varepsilon}\sqrt{\sigma_t \mu_i}$ the eigenvalues write

$$\lambda_1 = \sqrt{\frac{7}{3}}(\varepsilon\sigma_a + \frac{\sigma_s}{\varepsilon}), \qquad \lambda_2 = \sqrt{7}(\varepsilon\sigma_a + \frac{\sigma_s}{\varepsilon}),$$
$$\lambda_3 = \sqrt{p(\sigma_a,\sigma_s) + \sqrt{q(\sigma_a,\sigma_s)}}, \qquad \lambda_4 = \sqrt{p(\sigma_a,\sigma_s) - \sqrt{q(\sigma_a,\sigma_s)}},$$

where p and q are polynomial of degree 2.

- Since dim $Ker(R_1) \underset{\sigma_a \to 0}{\to} 1$, there exists one eigenvalue μ_j such that $\mu_j \underset{\sigma_a \to 0}{\to} 0$.
- There exist $\mathbf{q}_i(\mathbf{x})$ polynomial solution to the P_N model (15) when $\sigma_a = 0$. The functions $\mathbf{q}_{2k}(\mathbf{x})$ and $\mathbf{q}_{2k+1}(\mathbf{x})$ are polynomial functions of degree k and can be construct using recurrence formula.
- There exists $\tilde{\mathbf{e}}_1, \ldots, \tilde{\mathbf{e}}_{2n+1}$ linear combinations of the functions $(\mathbf{e}_j)_1, \ldots, (\mathbf{e}_j)_{2n+1}$ such that $\tilde{\mathbf{e}}_{2k} \to \mathbf{q}_{2k}(\mathbf{x})$ and $\tilde{\mathbf{e}}_{2k+1} \to \mathbf{q}_{2k+1}(\mathbf{x})$.

Example

For the P_1 model the first functions $\mathbf{q}_i$ write

$$\mathbf{q}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{q}_2 = \begin{pmatrix} -\frac{\sigma_s}{\varepsilon^2} x \\ 1 \\ 0 \end{pmatrix}, \mathbf{q}_3 = \begin{pmatrix} -\frac{\sigma_s}{\varepsilon^2} y \\ 0 \\ 1 \end{pmatrix}, \quad \ldots$$

P_3 : test case

- We consider a one dimensional test on the two dimensional P_3 model.
- Consider $\Omega = [0,2] \times [0,0.1]$, $\sigma_a(x,y) = 2 \times \mathbf{1}_{[0,1]}(x)$, $\sigma_s(x,y) = 100 \times \mathbf{1}_{[1,2]}(x)$.
- For the boundary condition we take $A^- \mathbf{u} = \delta_0(x)$.
- For each solutions we take 4 directions perpendicular to the edges. This give a total of 16 basis functions.

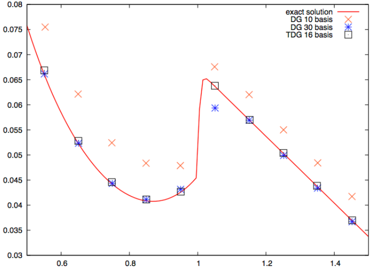
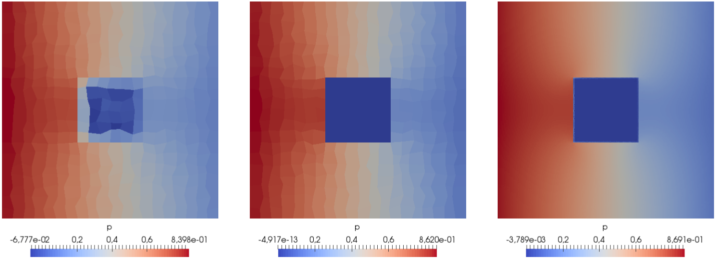


Figure: One dimensional view of the scalar flux, zoom on $[0.5,1.5] \times 0.05$. Comparison between the DG and TDG method. Random mesh with 20×5 cells.

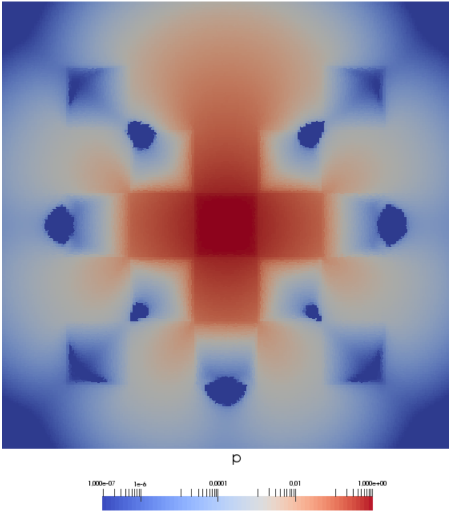
$P_3/2D : \mathbf{u} \in \mathbb{R}^{10}$



DG — Trefftz — ref. solution

Implementation C++ code by G. Morel :
- P1/P3, 1D/2D, stationary/non stationary,
- Viennagrid (mesh manager) : meshes $\geq 320 \times 320$ for P1, meshes until 160×160 for P3,
- linear solver : Eigen/Trilinos, ...

On going : $P3$ for the Brunner test



A conclusion

- **Pros** :
  - TDG incorporate a priori knowledge in the basis functions.
  - Easy to incorporate in DG codes since one only needs to change the basis functions and use ad-hoc quadratures.
  - Often need less DOF than DG to reach a given accuracy.
  - Outperforms DG for problems with boundary layers.
- **Cons** :
  - The practical calculation of the basis functions adds to the computational burden.
  - Poor linear independence of the basis functions.

Morel + D. + Buet : TDG method for Friedrichs systems : application to the P_1 model, HAL 2017.