

# MÉTHODE DÉTERMINISTE EF POUR LA RÉOLUTION DE L'ÉQUATION DE TRANSFERT RADIATIF

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M. A. Badri, P. Jolivet, B. Rousseau, and Y. Favennec

Workshop mathématiques pour la neutronique  
30 novembre

- neutron transport
- combustion
- optical tomography
- laser radiation...

## Propagation of photons

- absorption
- scattering
- emission

## Spectral radiance

- space coordinates  $(x, y, z)$
- solid angles  $(\theta, \psi)$
- wavelength  $\lambda$
- time  $t$

$\implies$  7D problem

$$\frac{1}{c} \frac{\partial I_\lambda}{\partial t} + \vec{s} \cdot \nabla I_\lambda + (\kappa + \sigma_s) I_\lambda = \sigma_s \oint_{\Omega} \varphi \cdot I_\lambda d\Omega + \kappa B_\lambda$$

$$(\vec{s} \cdot \nabla + (\kappa + \sigma_s)) I = \sigma_s \int_{\Omega} \varphi \cdot I \, d\Omega + \kappa B$$

- $\vec{s} = [\sin \theta \cos \psi \quad \sin \theta \sin \psi \quad \cos \theta]^T$ ,
- $\kappa$  absorption coefficient
- $\sigma_s$  scattering coefficient
- $\varphi$  phase scattering function
- $B$  black body emissivity function

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# STEADY-STATE MONOCHROMATIC RTE

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Deterministic methods:

- FVM (element-wise conservativity)
- FEM (flexibility)

Statistical methods:

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DG also possible [Le Hardy, Favennec, Rousseau 2016]

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- quadrature rules
- surrogate angular mesh

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## Semi-discretized RTE

$\forall m \in \llbracket 1; N_a \rrbracket$ :

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# NUMERICAL SCHEMES IN ANGLES

Discretization of the unit sphere:

- quadrature rules
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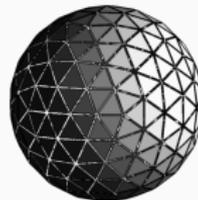
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$N_a = 80$



$N_a = 320$

# ANGULAR DECOMPOSITION

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Boundary conditions are not considered here.

Multiplying by a test function, we get  $\forall m \in \llbracket 1; N_a \rrbracket$ :

$$\int_{\mathcal{D}} \left[ (\vec{s}_m \cdot \nabla + (\kappa + \sigma_s)) l_m - \sigma_s \sum_{n=1}^{N_a} \omega_n \varphi_{m,n} \cdot l_n - \kappa B \right] (v + \gamma \vec{s}_m \cdot \nabla v) = 0$$

Or in more compact form,  $\forall m \in \llbracket 1; N_a \rrbracket$ :

$$a_m^{(d)}(l_m, v) + \sum_{n=1}^{N_a} a_{m,n}(l_n, v) = L_m(v)$$

# MATRICIAL FORMULATION

Given a FE space  $\mathcal{V}_h$  with  $N_h$  d.o.f., we end up with:

$$\begin{bmatrix} A_1^{(d)} + A_{1,1} & A_{1,2} & & A_{1,N_a} \\ A_{2,1} & A_2^{(d)} + A_{2,2} & & A_{2,N_a} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N_a,1} & A_{N_a,2} & & A_{N_a}^{(d)} + A_{N_a,N_a} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N_a} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{N_a} \end{bmatrix}$$

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To do (from a distributed-memory parallelism POV)

- assemble the matrix efficiently
- find an appropriate preconditioner

# DISTRIBUTED ASSEMBLY

Domain decomposition in angles

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Domain decomposition in angles (for  $N_a = 4$ )

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↔ with PU #0

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What happens with more PU? #4, #5, #6, and #7.

# DISTRIBUTED ASSEMBLY

## Redistribution

- traditional solvers: up to  $N_a$  PU
  - our method: up to  $k \cdot N_a$  PU, with  $k = 1, \dots, N_a$
- $\implies N_a = 320$ , up to 102,400 PU

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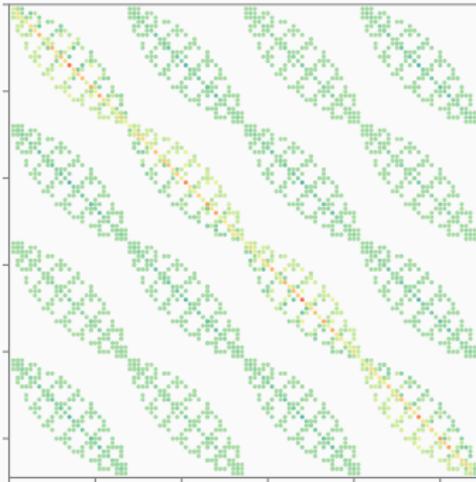
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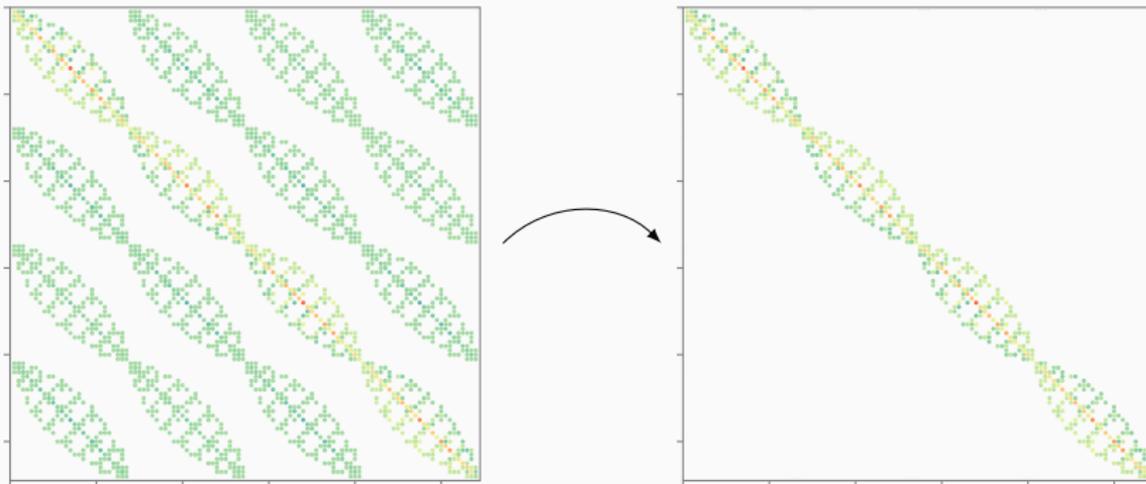
# PRECONDITIONING

- unsymmetric
- mixed dense-sparse structure
- large



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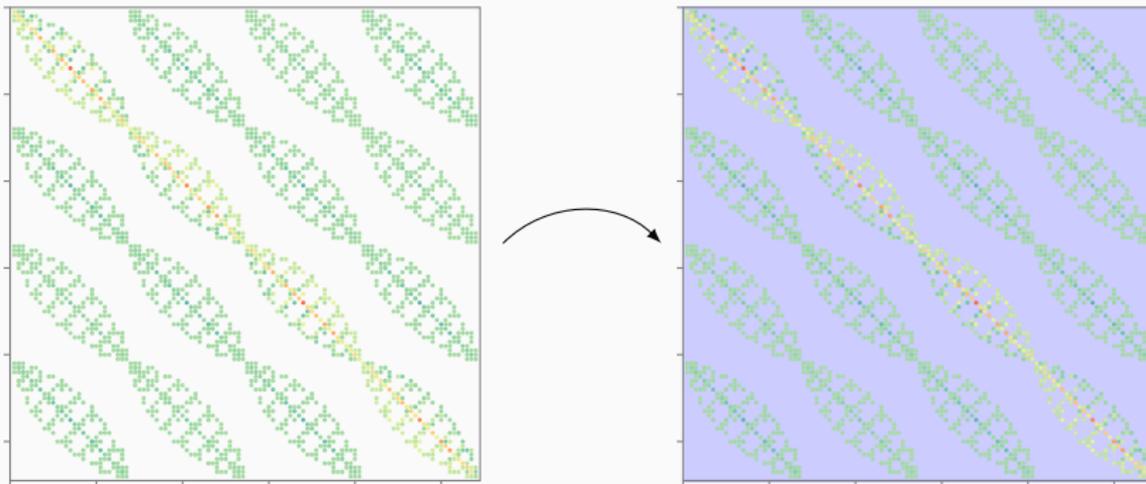
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Block Jacobi with  
at most  $N_a$  blocks

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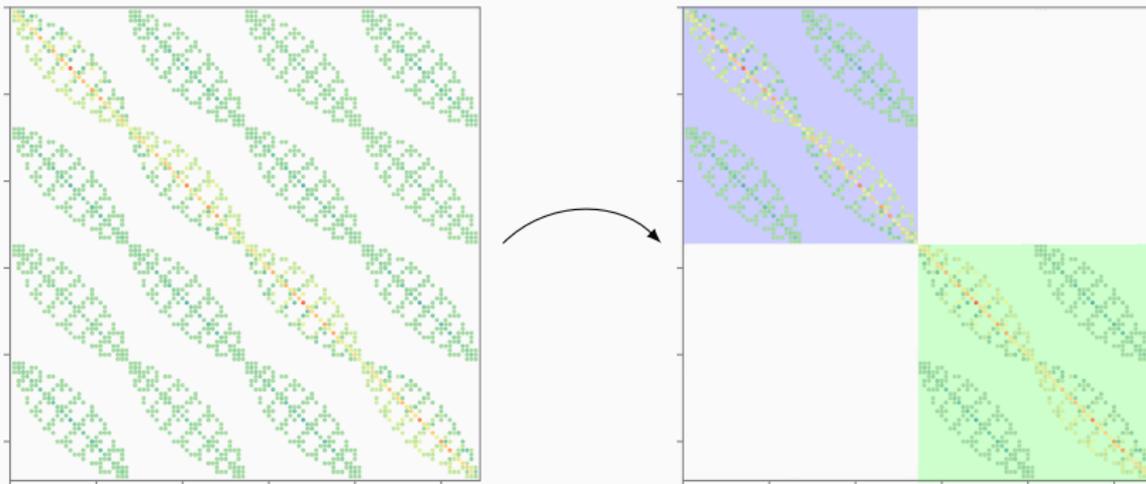
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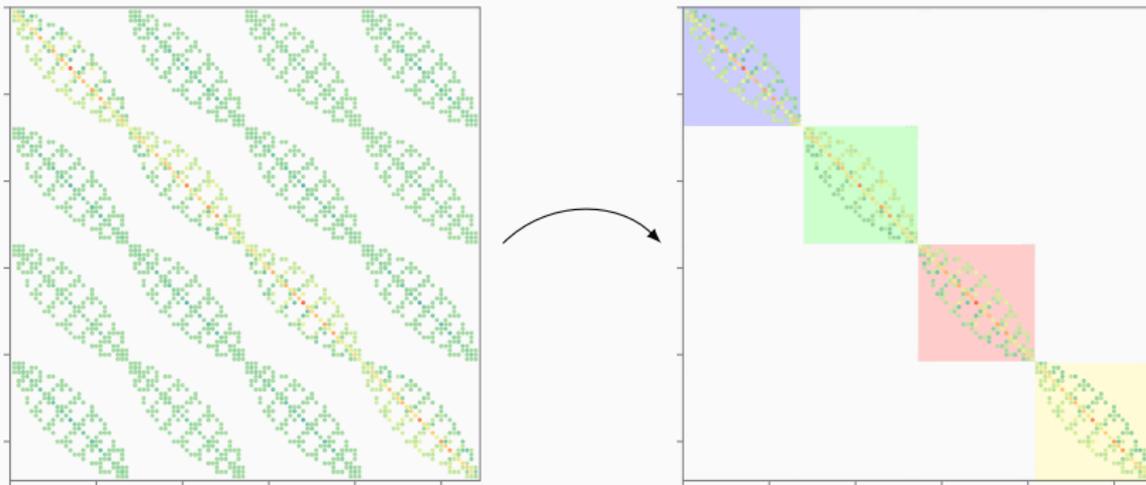
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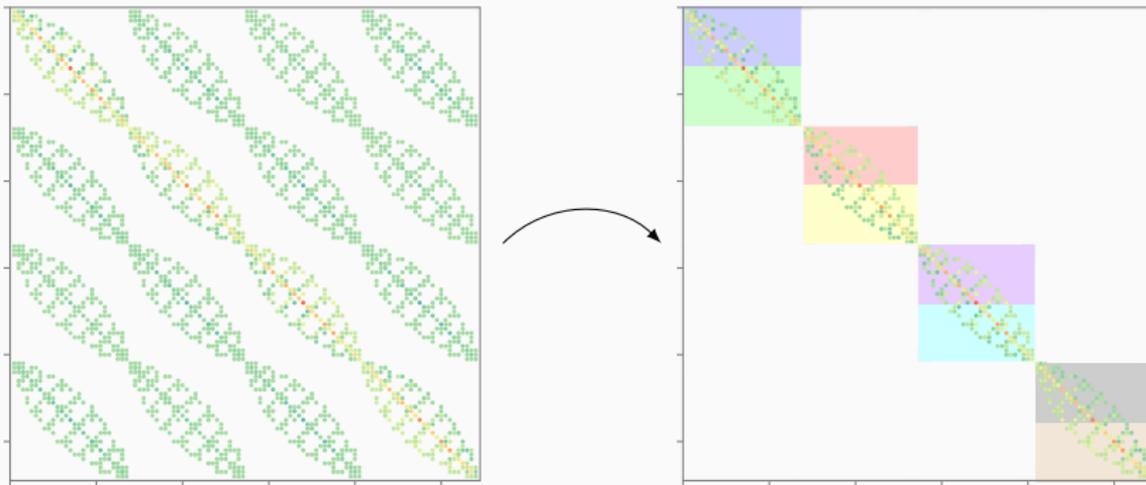
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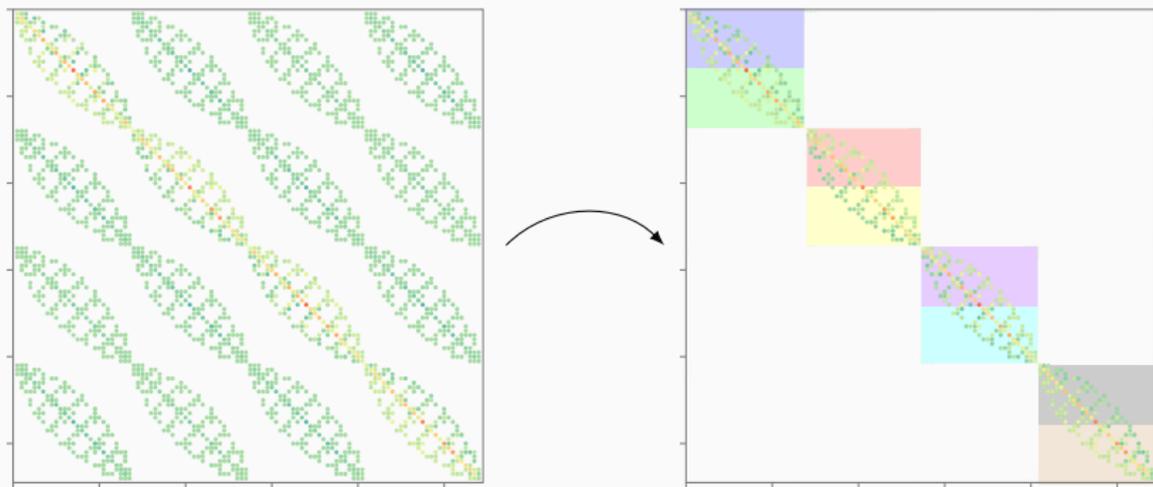
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$\implies$  `-pc_type bjacobi -pc_bjacobi_blocks  $N_a$`

# DOMAIN DECOMPOSITION

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Given a FE space  $\mathcal{V}_h^{N_a} = \mathcal{V}_h \times \mathcal{V}_h \times \cdots \times \mathcal{V}_h$ :

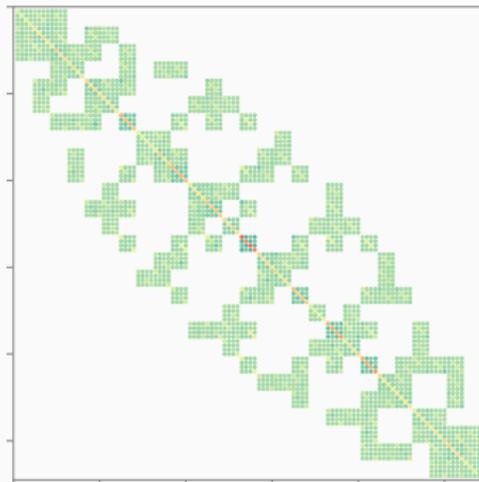
$$\int_{\mathcal{D}} \left[ \left( \vec{\mathbb{S}} \cdot \nabla + (\kappa + \sigma_s) \right) \mathbb{I} - \Phi : \mathbb{I} - \kappa B \mathbb{1} \right]^T \left( \mathbb{V} + \gamma \vec{\mathbb{S}} \cdot \nabla \mathbb{V} \right) = 0,$$

with

$$\Phi = \sigma_s \begin{bmatrix} \omega_1 \varphi_{1,1} & \omega_2 \varphi_{1,2} & & \omega_{N_a} \varphi_{1,N_a} \\ \omega_1 \varphi_{2,1} & \omega_2 \varphi_{2,2} & & \omega_{N_a} \varphi_{2,N_a} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1 \varphi_{N_a,1} & \omega_2 \varphi_{N_a,2} & & \omega_{N_a} \varphi_{N_a,N_a} \end{bmatrix}$$

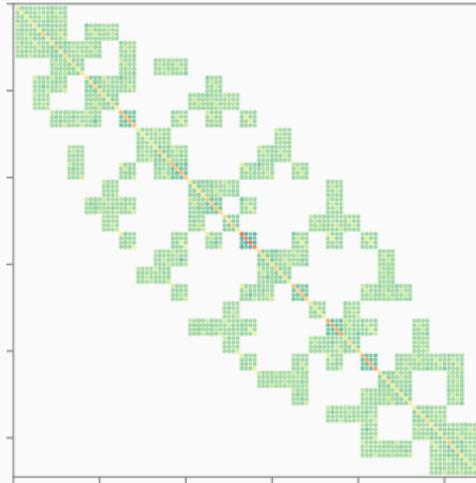
# MATRICIAL FORMULATION

$\implies$  global linear system of order  $N_a \times N_h$



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- sparse matrix with dense blocks of order  $N_a$
- number of nonzeros grows quadratically with  $N_a$

## Tensor products

- no need to deal with a huge variational formulation
- only assembles elementary matrices
- compute the global matrix using tensor products

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$$(\mathbb{I}, \mathbb{V}) \in \mathcal{V}_h^{N_a} \times \mathcal{V}_h^{N_a} \quad \left[ \left( \vec{\mathbb{S}} \cdot \nabla + (\kappa + \sigma_s) \right) \mathbb{I} \right. \\ \left. - \Phi : \mathbb{I} - \kappa B \mathbb{I} \right]^T (\mathbb{V} + \gamma \vec{\mathbb{S}} \cdot \nabla \mathbb{V})$$

$$(l, v) \in \mathcal{V}_h \times \mathcal{V}_h \quad \begin{array}{l} \frac{\partial l}{\partial x} v, \frac{\partial l}{\partial y} v, \frac{\partial l}{\partial z} v, l \cdot v, \frac{\partial l}{\partial x} \frac{\partial v}{\partial x}, \frac{\partial l}{\partial y} \frac{\partial v}{\partial y}, \frac{\partial l}{\partial z} \frac{\partial v}{\partial z}, \\ \frac{\partial l}{\partial x} \frac{\partial v}{\partial y}, \frac{\partial l}{\partial x} \frac{\partial v}{\partial z}, \frac{\partial l}{\partial y} \frac{\partial v}{\partial z} \end{array}$$

## Dealing with large dense blocks

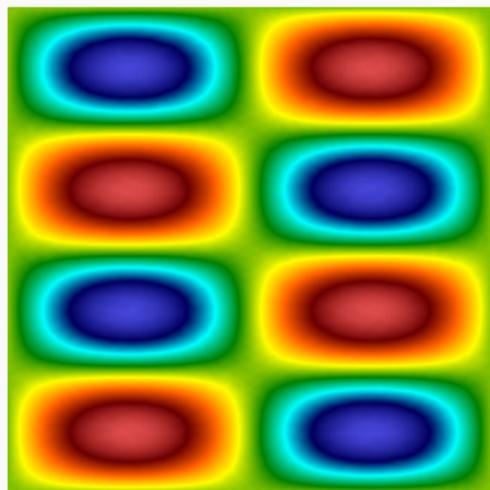
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  - efficient smoothers for MG
- 
- for angular decomposition, no long-range interaction
  - “pruned” matrix by removing off-diagonal elements of dense blocks

## NUMERICAL RESULTS

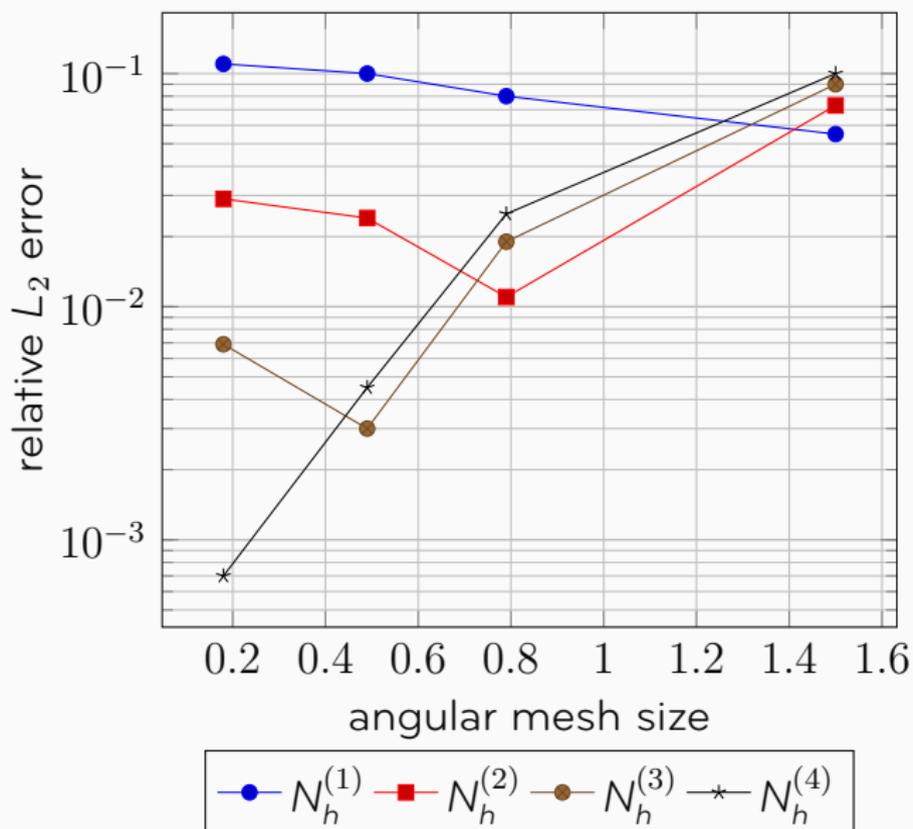
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$$D = 2\pi (1 + \sin(2\pi x) \sin(4\pi y))$$

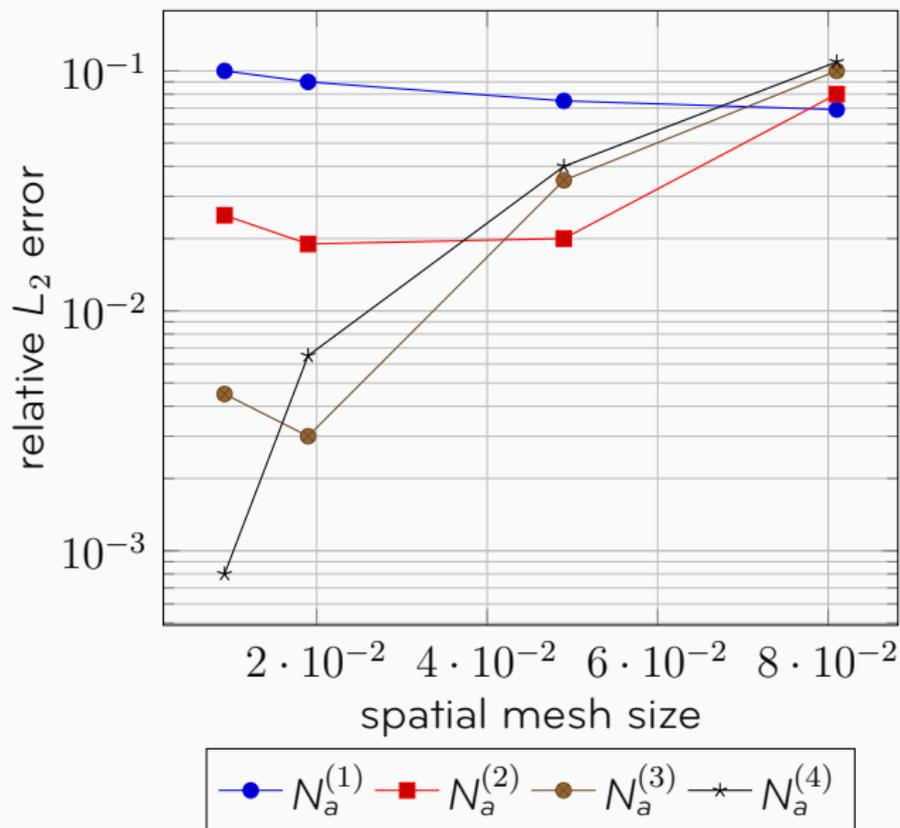
# MANUFACTURED SOLUTION

Angular mesh refinement

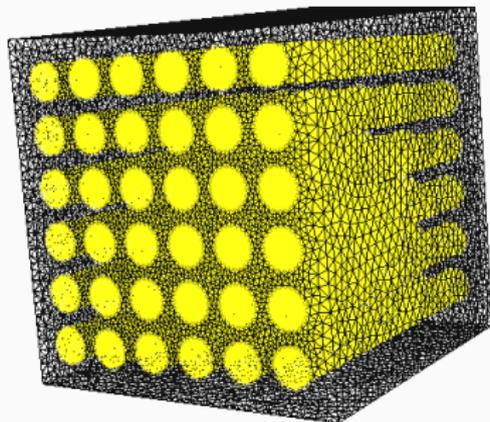


# MANUFACTURED SOLUTION

## Spatial mesh refinement



- Henyey–Greenstein phase scattering function  $\varphi$
- semi-transparent ( $\omega = 0.9$ ) material with absorbing ( $\omega = 0.09$ ) inclusions
- MUMPS used as the direct distributed solver
- Curie@TGCC



# LARGE-SCALE EXPERIMENTS

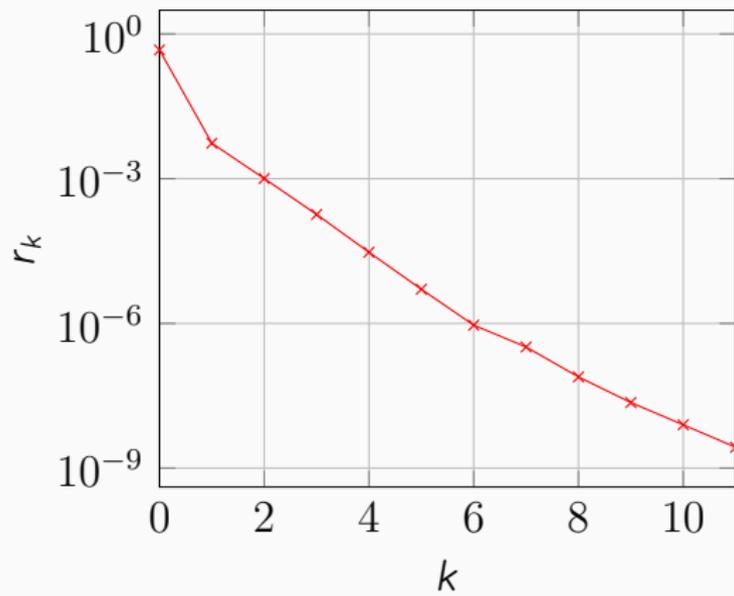
## Angular decomposition

- $N_a = 320$
- 6,400 MPI processes
- matrix assembly: 39.93s
- matrix redistribution: 3.98s
- setup: 37.83s
- solve: 104.92s (60.98s for matrix-vector products)

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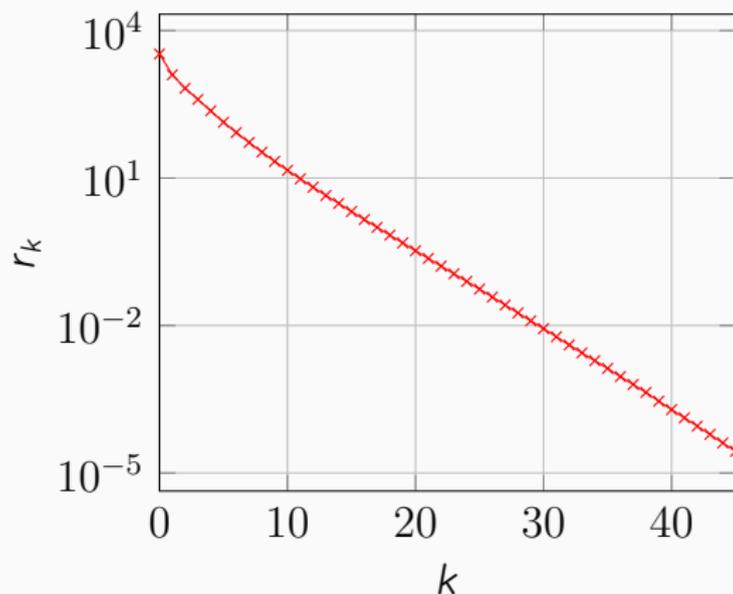
- $N_a = 512$
  - 8,192 MPI processes
  - matrix assembly: 5.96s
  - setup: 17.42s
  - solve: 39.45s (34.9s for MV products)
- $\implies 10^{12}$  nnz distributed over  $124 \cdot 10^6$  blocked rows

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- domain decomposition with vectorial FE
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